Optimizing Algebraic Programs

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Tim Sheard* & Leonidas Fegaras†
Oregon Graduate Institute of Science & Technology
sheard@cse.ogi.edu fegaras@cse.ogi.edu

Abstract

This paper considers a programming language where all control is encoded in algebras and combinators over algebras. This language supports higher levels of abstraction than traditional functional languages and is amenable to calculation based optimization. Three well known transformations are illustrated. Each one, requiring varying levels of insight and creativity over ordinary functional programs, can be fully automated in an algebraic language. The algorithm encoding these transformations is presented. This algorithm is an improvement over our previous work since it works over a richer, more expressive language, encodes more transformations, and is more efficient.

1 Introduction

We have developed a programming style we call *algebraic programming* because of its reliance on algebras and combinators for encoding control. Algebraic programs provide two advantages over traditional functional programs. First, they provide a mechanism for abstracting over type constructors, i.e. it is possible to encode algorithms which work over *any* datatype definition. Second, such algorithms have generic theorems, which make it possible to build semantic based optimizations. Such optimizations are amenable to automation, whereas languages which allow arbitrary recursive programs lack the structure necessary for this kind of automation without expensive analysis.

The contributions of this paper are several. First we extend our earlier work[5, 6, 17] by embedding our optimizations in a richer language, and describe an improved algorithm for computing them. Our previous work focused on a restricted language which has now been extended to include all the features of a modern functional programming language. The improved algorithm works over the entire extended language while the original algorithm worked only on a syntactically identifiable subset of the restricted language.

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Second, we report on an actual implementation[12] which includes a user oriented front-end as well as our optimizing back-end, and a compiler. Experiments with our implementation illustrate that is feasible, practical, and beneficial to program in an algebraic style.

Third, we illustrate that it is possible to extend our techniques by the use of optimization algorithms based upon additional generic theorems. We describe three transformations which heretofore either required human intervention in the form of "eureka" steps or generalization choices, or time consuming search based analysis. The explicit structure of algebraic programs makes these transformations evident by simple inspection or reduction based techniques. In fact, we have developed and implemented an algorithm which utilizes just these three transformations and which manages to capture many well known optimizations. It is our belief that additional generic strategies can strengthen our algorithm.

2 Three Transformations

In the this section we quickly outline three transformations in terms of traditional functional programming. The first two were first described over traditional functional programs by Burstall and Darlington[1], the third is a generic construction of distributive laws we believe is new in any context. In subsequent sections we describe how algebraic programs make these transformations evident.

2.1 Simultaneous Traversal of a Single Structure

Two algorithms which both traverse a single structure can often be fused into a single traversal. This technique is a generalization of loop fusion to arbitrary data structures. As an example consider computing the length and sum of a list simultaneously as in the example $avg\ x = (sum\ x)/(len\ x)$. Given the definitions

$$\begin{array}{lcllcl} len\ \text{nil} & = & 0 & sum\ \text{nil} & = & 0 \\ len\ (\text{cons}(x, xs)) & = & 1 + (len\ xs) & sum\ (\text{cons}(x, xs)) & = & x + (sum\ xs) \end{array}$$

we proceed by introducing a function that computes both values: $h\ x = (len\ x,\ sum\ x)$, and by instantiating this function on the two cases which construct legal lists

$$\begin{array}{lcl} h\ nil & = & (len\ 0, sum\ 0) \\ & = & (0, 0) \\ \\ h\ (\text{cons}(x, xs)) & = & (len(\text{cons}(x, xs)), sum(\text{cons}(x, xs))) \\ & = & (1 + len\ xs, x + sum\ xs) \end{array}$$

At this point we need to generalize two of the terms in order to complete the transformation. It is this step that often requires human intervention.

$$\begin{array}{rcl} h\ (\text{cons}(x, xs)) & = & (1 + u, x + v)\ \textbf{where}\ (u, v) = (len\ xs, sum\ xs) \\ & = & (1 + u, x + v)\ \textbf{where}\ (u, v) = h\ xs \end{array}$$

Thus completing the definition of the function that simultaneously computes both the length and sum of a list.

2.2 Deforestation

Functional programming encourages the definition of complex functions as the composition of simpler ones. This often introduces an unnecessary intermediate data structure. For example consider computing: $len(map\ f\ x)$ where

$$\begin{array}{lcl} map\ f\ \text{nil} & = & \text{nil} \\ map\ f\ (\text{cons}(x, xs)) & = & \text{cons}(f\ x,\ map\ f\ xs) \end{array}$$

Computing $(map\ \ f\ \ x)$ produces an intermediate list which is then consumed by len. We proceed in much the same manner as for simultaneous traversal, introducing a new function h, and using the definitions to "arrange" a *recursive* call to h.

$$\begin{array}{lcl} h\ x & = & len(map\ f\ x) \\ \\ h\ \text{nil} & = & len(map\ f\ \text{nil}) \\ & = & len(\text{nil}) \\ & = & 0 \\ \\ h\ (\text{cons}(x, xs)) & = & len(map\ f\ (\text{cons}(x, xs))) \\ & = & len(\text{cons}(f\ x,\ map\ f\ xs)) \\ & = & 1 + (len(map\ f\ xs)) \\ & = & 1 + h\ xs \end{array}$$

In this example "arranging" for the recursive call to appear was easy. In general this is not the case.

2.3 Distributive Law Generation

Arranging for the recursive call to appear often depends upon knowledge of distributive laws. For example consider $h\ x\ =\ len(rev\ x)$ where

$$\begin{array}{lcl} \text{nil} \mathbin{@} y & = & y \\ (\text{cons}(x, xs)) \mathbin{@} y & = & \text{cons}(x,\ xs \mathbin{@} y) \end{array} \qquad \begin{array}{lcl} rev\ \text{nil} & = & \text{nil} \\ rev\ (\text{cons}(x, xs)) & = & (rev\ xs) \mathbin{@} (\text{cons}(x, \text{nil})) \end{array}$$

For the case $x\ =\ \text{cons}(y, ys)$ we proceed

$$\begin{array}{lcl} h\ (\text{cons}(y, ys)) & = & len(rev\ (\text{cons}(y, ys))) \\ & = & len((rev\ ys) \mathbin{@} (\text{cons}(y, \text{nil}))) \end{array}$$

At this point we need the distributive law: $len(x \mathbin{@} y)\ =\ (len\ x) + (len\ y)$ to complete the transformation.

$$\begin{array}{lcl} & = & len(rev\ ys) + len(\text{cons}(y, \text{nil})) \\ & = & len(rev\ ys) + 1 + len(\text{nil}) \\ & = & len(rev\ ys) + 1 + 0 \\ & = & h(ys) + 1 \end{array}$$

Algebraic programs provide the means to generate this law (and many others) on demand, thus somewhat alleviating the need for a library of laws, which no matter how large, will often be incomplete.

3 Algebraic Programming in ADL

In this section we describe our implementation of algebraic programming we call ADL (*Algebraic Design Language*).

3.1 Signatures

ADL departs significantly from functional programming languages such as SML by providing declarations of signatures that define varieties of structure algebras, not simply datatypes. An algebraic signature consists of a finite set of operator names, together with the type of the domain of each operator. The co-domain of an operator is the carrier type for each particular algebra. For example:

```
signature List{type c; $list(a)/c = {$nil, $cons of (a * c)}}
signature Tree{type c; $tree(a)/c = {$tip, $fork of c * a * c}}
```

The *list*-sorted algebras have a signature parametric on a type represented by the variable a. The type variable c is used as the name of the carrier type. The signature consists of a pair of operator names, with typing: $\$nil : c, \$cons : a \times c \to c$. Note that the domain of each operator is explicit in the signature, and the codomain of each operator is implicitly given by the carrier.

Defining a signature in ADL causes several types and functions to be defined automatically. One is the *freely constructed* type which would correspond to a datatype definition in language like ML. Several other functions and types are defined automatically as well. The induced types and functions of the list signature would be declared as follows in Standard ML:

```
datatype 'a list = nil | cons of ('a *  'a list);
datatype ('a,'c) E$list = $nil | $cons of ('a * 'c);
fun E$nil  (f_a,f_c) () = ();
fun E$cons (f_a,f_c) (x,y) = (f_a x, f_c y);
```

All the functions and types above become accessible to the programmer by simply writing the signature declaration. Note that `list` is the usual, recursive, freely constructed type, where the carrier `c` has been replaced with the type being defined (`'a list`), and `cons` and `nil` are the free constructors. The type `E$list` is a non-recursive type which has one extra type parameter corresponding to the carrier of the signature. Constructors of this non-recursive type have the same name as the signature operators, e.g. `$cons, $nil` (different from the free constructors `cons` and `nil`) and are called *algebra operators*. To construct the definitions of the functions `E$nil` and `E$cons` from a signature declaration we are guided by the types of the domains of the corresponding signature operators `$nil` and `$cons`. This can be done by lifting the `*`

operator to work on functions as follows: `(f * g)(x,y) = (f x, g y)`. This construction guarantees that the functions `E$nil` and `E$cons` are functors. Like any functor they preserve identities and compositions:

$$\begin{array}{lcl} E\$cons(id, id) & = & id \\ E\$cons(f, g) \circ E\$cons(h, k) & = & E\$cons(f \circ h, g \circ k) \end{array}$$

These functions will play an important part in the sequel. Similar types and functions are induced for `tree` and other signatures.

3.2 Algebras

A concrete algebra is specified by a structure that contains bindings for the carrier type and for each operator of the algebra. Examples for the `list` and `tree` algebras include:

```
List{c := int; $list{$nil := 0, $cons := \(x,y) 1+y}}
List{c := int; $list{$nil := 0, $cons := \(x,y) x+y}}
Tree{c := list(a); $tree{$tip := nil, $fork := \(x,y,z) x @ [y] @ z}}
```

Given the type of the carrier, the assignment of functions to the operators of the signature must be consistently typed.

3.3 Combinators

ADL encodes control in a standard set of combinators that take an algebra as an argument and return a function which is a free algebra morphism (a function over the freely constructed type). The combinators include *map*, *reduction* (fold,catamorphism), *primitive recursion* (paramorphism), *derive* (unfold, anamorphism), and *hom* (hylomorphism), as well as the duals of these combinators (*cohom* etc.) and mechanism for interpreting all these morphisms in an arbitrary monad. We will not discuss these additional mechanisms here[12].

When a combinator is applied to an algebra specification the returned morphism obeys a set of recursive equations particular to that algebra. For example the `red` combinator applied to a `list` algebra obeys:

$$\begin{array}{lllcl} \text{red}[list]\ List\{c; \$nil, \$cons\} & \text{nil} & & = & \$nil \\ \text{red}[list]\ List\{c; \$nil, \$cons\} & (\text{cons}\,(x, y)) & & = & \$cons\,(x,\ \text{red}[list]\ List\{c; \$nil, \$cons\}\ ) \end{array}$$

Note that a combinator cannot be typed in an ML-like language since it is parametric over algebras of any signature*. In general, the recursive equation a combinator obeys can only be described in a meta-language (since ADL itself has no recursion). In the meta-language this can be done using the induced functors, $E\$c_i$. Given any constructor c_i with type $t_i \to T$ the reduction combinator obeys:

$$\text{red}[T]\ T\{\$c_i := f_i\}\quad (c_i\ x) \quad = \quad f_i\ (E\$c_i\ (id, \text{red}[T]\ T\{\$c_i := f_i\})\ x)$$

Some example definitions in ADL of functions using `red` are:

*In ML we could represent an algebra by a tuple of functions. A single function `red` that takes any algebra as input and returns a reduction for that algebra cannot be typed.

```
val sum = red[list] List{c:=int; $nil := 0, $cons := (op +) };
val len = red[list] List{c:=int; $nil := 0, $cons := \(x,y) y+1 };
val map f = red[list] List{c:=list(a); $nil := nil, $cons := \(x,y) cons(f x,y) };
val flatten = red[tree] Tree{c:=list(a); $tip:=nil, $fork:=\(l,x,r) l @ [x] @ r};
```

The `hom` combinator applied to a *list* algebra with carrier c and a splitting function $P : \gamma \rightarrow E\$list(\alpha, \beta)$ returns a morphism with type $\gamma \rightarrow c$ which obeys the recursive equation:

$$
\begin{array}{l}
\text{hom}[list]\ List\{c;\ \$nil := f,\ \$cons := g\}\ P\ x\ = \\
\textbf{case}\ P\ x\ \textbf{of} \\
\quad \$nil \Rightarrow f \\
\mid \$cons\ (a,b) \Rightarrow g(a,\ \text{hom}[list]\ List\{c;\ \$nil := f,\ \$cons := g\}\ P\ b)
\end{array}
$$

The function $(hom[T]\ T\{\ldots\}\ P\ x)$ recurses over the structure of T found in x which is induced by applying the splitting function P. The general equation can again be expressed in terms of the induced functors.

$$
\begin{array}{l}
\text{hom}[T]\ T\{\ldots, \$c_i := f_i, \ldots\}\ P\ (c_i\ x)\ = \\
\textbf{case}\ P\ x\ \textbf{of} \\
\quad \ldots \\
\mid \$c_i\ y \Rightarrow f_i(E\$c_i\ (id,\ \text{hom}[T]\ T\{\ldots, \$c_i := f_i, \ldots\}\ P)\ y) \\
\quad \ldots
\end{array}
$$

An example definition in ADL of a function using `hom` is the `upto` function.

```
val upto = \(m) \(n) (hom[List] {c:= List(int); $nil := nil, $cons := cons}
                          (\ x (if x>n then $nil else $cons(x,x + 1))) m );
```

A slightly more complicated example is quick-sort

```
val flatAlg = {c:=list(a); $tip := nil, $fork := \(l,x,r) l @ [x] @ r};
val Split = \(x) case x of
                   nil => $tip
                 | cons(x,y) => $fork(filter (<= x) y,x,filter (> x) y);
val quicksort = hom[tree] flatAlg Split;
```

In a sense the `hom` combinator is more general than the `red` and other combinators, since all of the other combinators can be expressed in terms of `hom`. For example

$$
\begin{array}{rcl}
\text{red}[T]\ T\{\ldots, \$c_i := f_i, \ldots\} & = & \text{hom}[T]\ T\{\ldots, \$c_i := f_i, \ldots\}\ out^T \\
 & & \textbf{where:}\ out^T \circ C_i\ =\ \$C_i
\end{array}
$$

Out^T is a particularly simple splitting function, it replaces the top-most free constructor with the its corresponding algebra operator from the type $E\$T$. This has particular importance for our transformation techniques, since the internal data structures representing programs manipulated by our algorithms need deal with only a single combinator. The user may use the simpler combinators which are translated by the compiler into `hom`.

4 The Promotion Theorem.

The promotion theorem for `red` describes the conditions under which the composition of a function g with a `red` can be expressed as another `red` [14, 15]. For a list algebra the theorem is given below.

$$\frac{\begin{array}{lcl}\phi_n() & = & g(f_n()) \\ \phi_c(a, g(r)) & = & g(f_c(a,r))\end{array}}{g(\text{red}[list]\{f_n, f_c\}\ x) \ = \ \text{red}[list]\{\phi_n, \phi_c\}\ x}$$

A similar theorem can be expressed for every signature. In general the promotion theorem can be given for `hom` as well as `red` and can be expressed in terms of the induced functors $E\$c_i$ for any signature:

$$\frac{\forall i : \ h_i \circ E\$c_i(id, g) \ = \ g \circ f_i}{g \circ (\text{hom}[T]\{\ldots \$c_i := f_i \ldots\}\ P) \ = \ \text{hom}[T]\{\ldots \$c_i := h_i \ldots\}\ P}$$

The promotion theorem, when used as a left to right rewrite rule, implements a form of fusion. To apply it, the functions, h_i, which meet the stated conditions of the premise must be found.

5 Three transformations for Algebraic Programs

In this section we describe how the structure of algebraic programs enables the three transformations described earlier.

5.1 Deforestation and The Normalization Algorithm

The Normalization algorithm is an effective algorithm for computing the h_i's of the promotion theorem. It based upon the algorithm of [17] extended in the following ways. First, it is based upon the `hom` promotion theorem rather than `red`, second it computes over a richer language, and terminates over the complete language rather than a syntactically identifiable subset. This algorithm automates deforestation [8, 20, 19] and fusion [2] for algebraic programs.

From the promotion theorems we know *only the property* that the h_i's should obey, not how to compute them. The following construction is the basis for the normalization algorithm. Given:

$$h_i \circ E\$c_i(id, g) \ = \ g \circ f_i \quad \text{the requisite property}$$

suppose there exists a function $\mathbf{inv}^g$ with property $g(\mathbf{inv}^g\ x) \ = \ x$. Then [†]:

$$\begin{array}{lcll} h_i \circ E\$c_i(id, g) \circ E\$c_i(id, \mathbf{inv}^g) & = & g \circ f_i \circ E\$c_i(id, \mathbf{inv}^g) & \\ h_i \circ E\$c_i(id, g \circ \mathbf{inv}^g) & = & g \circ f_i \circ E\$c_i(id, \mathbf{inv}^g) & \text{by functorality of } E\$c_i \\ h_i \circ E\$c_i(id, id) & = & g \circ f_i \circ E\$c_i(id, \mathbf{inv}^g) & \text{by property of } \mathbf{inv}^g \\ h_i & = & g \circ f_i \circ E\$c_i(id, \mathbf{inv}^g) & \text{by functorality of } E\$c_i \end{array}$$

[†]Since in reality $\mathbf{inv}^g$ may not exist, in the algorithm it only plays the role of a placeholder as shall see in the sequel.

The normalization algorithm works on the formula $g \circ f_i \circ E\$c_i(id, \mathbf{inv}^g)$; it attempts to push the g towards the $\mathbf{inv}^g$ so that they may cancel each other. In order for the algorithm to be effective, it must rely on no property of $\mathbf{inv}^g$ other than $g(\mathbf{inv}^g\ x) = x$, and remove all occurrences of $\mathbf{inv}^g$. For example consider

```
len(map f x) = len(red[list] {c:=list(a); $nil:=nil, $cons:=\(x,y) cons(f x,y)})
```

Setting up the equation for `nil` we begin:

```
$nil = len nil = 0
```

for `cons` we proceed:

```
$cons(z,zs) = len ( (\(x,y) cons(f x,y)) (E$cons(id,Inv(len)) (z,zs)) )
$cons(z,zs) = len ( (\(x,y) cons(f x,y)) (z, (Inv(len)) zs)) )
$cons(z,zs) = len ( cons(f z, (Inv(len)) zs) )
$cons(z,zs) = 1 + (len ((Inv(len)) zs) )
$cons(z,zs) = 1 + zs
```

The normalization algorithm is a reduction engine which carries out this process and which instantly recognizes illegal uses of $\mathbf{inv}^g$ by using exceptions. Its reduction rules are based upon β-reduction, reduction of combinators over freely constructed objects, and the promotion theorem. Given an inverse free term, it returns an equivalent term (possibly the same term). For simplicity the algorithm here is expressed in terms of `red`, though our actual implementation is based upon `hom`.

$N\ term = \mathbf{case}\ term\ \mathbf{of}$

$\mathbf{inv}^g$	$\Rightarrow$	$\mathbf{raise}\ inverse$	illegal use of inverse
v	$\Rightarrow$	v	variable
$(t_1, \ldots, t_n)$	$\Rightarrow$	$((N\ t_1), \ldots, (N\ t_n))$	tuple
$\lambda v\,.\,e$	$\Rightarrow$	$\lambda v\,.\,(N\ e)$	abstraction
$c_i\ \bar{x}$	$\Rightarrow$	$c_i\ (N\ \bar{x})$	construction
$(\lambda v\,.\,b)\ x$	$\Rightarrow$	$N(\text{Beta}\ v\ b\ x)$	β reduction
$g(\mathbf{inv}^g y)$	$\Rightarrow$	y	Success !!!
$g(\text{red}[T]\{\$c_i := f_i\}\ x)$	$\Rightarrow$	$\begin{cases} (\text{red}[T]\{\$c_i := h_i\}\ x) \\ \quad \text{where } h_i = \lambda \bar{y}\,.\,N(g(f_i(E\$c_i\ (id, \mathbf{inv}^g)\ \bar{y}))) \\ \mathbf{handle}\ inverse \Rightarrow \\ \quad (Ng)(N(\text{red}[T]\{\$c_i := f_i\}\ x)) \end{cases}$	promotion theorem
$f\ x$	$\Rightarrow$	$(N\ f)\ (N\ x)$	normal application
$\text{red}[T]\{\$c_i := f_i\}\ (c_i\ x)$	$\Rightarrow$	$N(f_i(E\$c_i(id, \text{red}[T]\{\$c_i := f_i\})\ x))$	combinator reduction
$\text{red}[T]\{\$c_i := f_i\}\ x$	$\Rightarrow$	$\text{red}[T]\{\$c_i := (N\ f_i)\}\ (N\ x)$	

Where the variables $\bar{y}$ introduced in the promotion step are new variables. If in the promotion step, N fails to compute the h_i, this is signaled by the exception *inverse* and handled by normalizing the two pieces of the promotion step independently, thereby no longer introducing any inverse terms. A proof of correctness of the normalization algorithm can be found in a technical report[18].

5.2 Simultaneous Traversal and the Tupling Lemma

Algebraic programs make it easy to recognize situations where simultaneous traversal is applicable. *Any two* reductions over the *same* variable can be traversed simultaneously! For example consider the example:

$$(\mathit{len}\ x,\ \mathit{sum}\ x)\ =\\ (\mathrm{red}[\mathit{list}]\{\$\mathit{nil} := 0, \$\mathit{cons} := \lambda\,(x,y)\,.\,1+y\}\ x,\ \mathrm{red}[\mathit{list}]\{\$\mathit{nil} := 0, \$\mathit{cons} := \lambda\,(x,y)\,.\,x+y\}\ x)$$

This example transforms into

$$\mathrm{red}[\mathit{list}]\{\$\mathit{nil} := (0,0), \$\mathit{cons} := \lambda\,(x,(u,v))\,.\,(1+u, x+v)\}\ x$$

A general formula for this transformation is called the Tupling lemma[10]. It can be stated for `red` as follows:

$$(\,\mathrm{red}[T]\{\ldots,\$c_i := f_i,\ldots\}\ x\,,\,\mathrm{red}[T]\{\ldots,\$c_i := g_i,\ldots\}\ x\,)\ =\\ \mathrm{red}[T]\{\ldots,\langle\ (f_i \circ (E\$c_i(\mathit{id},\mathit{first})), (g_i \circ (E\$c_i(\mathit{id},\mathit{second}))\ \rangle,\ldots\}\ x$$

where $\langle f,g\rangle\ x\ =\ (f\ x,\ g\ x)$ and $\mathit{first}(x,y)\ =\ x$ and $\mathit{second}(x,y)\ =\ y$. An analogous formula for `hom` is also useful.

$$(\,\mathrm{hom}[T]\{\ldots,\$c_i := f_i,\ldots\}\ P\ x\,,\,\mathrm{hom}[T]\{\ldots,\$c_i := g_i,\ldots\}\ P\ x\,)\ =\\ \mathrm{hom}[T]\{\ldots,\langle\ (f_i \circ (E\$c_i(\mathit{id},\mathit{first})), (g_i \circ (E\$c_i(\mathit{id},\mathit{second}))\ \rangle,\ldots\}\ P\ x$$

The explicit nature of control in algebraic programs make it possible to recognize and combine two traversals over a single data structure into a single traversal by inspection.

5.3 Generating Distribution Laws for Zero Replacements

Chin [2] relates how the use of laws may improve the deforestation process. We illustrated this in Section 2.3. The explicit structure of algebraic programs makes it possible to calculate some of the necessary laws on demand. In this section we describe how this may be done for a large class of programs. We believe that similar constructions can be found for other classes as well. A common function over an arbitrary type T which has a unique zero constructor, Z, (a nullary constructor like *nil* for list) is defined by:

$$(Zr^T\ y)\ x\ =\ \mathrm{red}\,[T]\,\{T;\ \ldots, \$Z = y,\ \ldots\ , \$C_i = C_i,\ \ldots\}\ x$$

Here the operator for the nullary constructor is assigned y as its meaning, and every other operator is assigned its corresponding free constructor. We call this function a *zero replacement function*. Zr^T has type $T \to T \to T$. A function $h(x,y) = Zr\ y\ x$ is associative ($h(w,h(x,y))\ =\ h(h(w,x),y)$), and has the zero, Z, for both a left and right identity ($h(x,Z)\ =\ x$ and $h(Z,y)\ =\ y$) [13]. Recognize that Zr^{list} is the list append operator, and that Zr^{nat} is natural number addition.

We postulate that zero replacement functions have an additional important property: $\forall f \epsilon$ red[T], $\exists g : f \circ (Zr\ y) = g \circ f$. We make this postulation since the Normalization algorithm provides an effective method to compute g.

For example consider the term $length(x \ @ \ y)$. Both *length* and @ are reductions so there must be a function g such that $length(x \ @ \ y) = g(length\ x)$[‡]. Since *length* is a reduction suppose that g is a reduction as well and can be expressed as red[*nat*] $\{\$Zero := \mathbf{m}, \$Succ := \mathbf{n}\}$, where **m** and **n** are arbitrary unknown functions. By normalizing both $length(x \ @ \ y)$ and red[*nat*] $\{\$Zero := \mathbf{m}, \$Succ := \mathbf{n}\}$ $(length\ x)$ we obtain two reductionss which compute the same value. Matching these terms against each other we find bindings for **m** and **n**, thus effectively computing g. For example, $length\ (x \ @ \ y)$ normalizes to:

$$\begin{array}{l} \text{red}\ \ [list] \\ \quad \{\$nil := \boxed{\text{red}[list]\ \{\$nil := Zero,\ \$cons := \lambda\,(y0, y1)\,.\,(Succ\ y1)\ \}\ y}, \\ \quad\ \$cons := \lambda\,(y3, y2)\,.\,(\boxed{Succ}\ y2)\} \end{array}$$

And the term $g(length\ x) = \text{red}[nat]\{\$Zero := m,\ \$Succ := n\}\ (length\ x)\}$ normalizes to:

$$\begin{array}{l} \text{red}\ \ [list] \\ \quad \{\$nil := \boxed{\mathbf{m}}, \\ \quad\ \$cons := \lambda\,(y4, y5)\,.\,(\boxed{\mathbf{n}}\ y5)\}\ x \end{array}$$

Matching[§] the two terms we find bindings for **m** and **n**. This is illustrated by the boxed terms in the diagram above. Recognize that **m** computes $length\ y$ thus g is red[*list*] $\{\$nil := length\ y,\ \$cons := \lambda\,(x, y)\,.\,Succ\ y\}$ which can be recognized as $g = \lambda\,x\,.(length\ y) + x$. Thus we have effectively computed the law $length(x \ @ \ y) = g(length\ x) = (length\ y) + (length\ x)$.

This technique allows us to calculate such laws as

$$\begin{array}{lcl} map\ f\ (x \ @ \ y) & = & (map\ f\ x) \ @ \ (map\ f\ y) \\ length\ (x \ @ \ y) & = & (length\ x)\ +\ (length\ y) \\ rev\ (x \ @ \ y) & = & (rev\ x) \ @ \ (rev\ y) \\ w * (x\ +\ y) & = & (w * x)\ +\ (w * y) \end{array}$$

This makes possible the automatic fusion of unsafe terms[2], such as $map\ f\ (rev\ x)$ without the need of any additional laws as was illustrated in Section 2.3 above.

6 Second Order Extensions

A second order reduction, red[T]$(\bar{f})\ x$, is a reduction which returns a function. Under certain circumstances the composition of a function g with such a reduction can be cast as another reduction composed with g.

[‡]Note that g is probably dependent on y.

[§]Unification where variables, **m** and **n** in this case, may appear in only one of the terms.

Theorem 1 (Second-order Promotion Theorem)

$$\frac{\forall i:\ (E_i(\lambda r\,.\,r\circ g)\ \bar{r})\ =\ (E_i(\lambda s\,.\,g\circ s)\ \bar{s})\ \ \Rightarrow\ \ (h_i\ \bar{r})\circ g\ =\ g\circ(f_i\ \bar{s})}{g\circ(\mathrm{red}[T](\bar{f})\,x)\ =\ (\mathrm{red}[T](\bar{h})\,x)\circ g}$$

proof: Let $x\ =\ C_i\ \bar{x}$, for arbitrary constructor C_i, then

$$\begin{array}{lll} & g\circ(\mathrm{red}[T](\bar{f})\,(C_i\ \bar{x})) & \\ =\ & g\circ(f_i\,(E_i(\mathrm{red}[T](\bar{f}))\ \bar{x})) & \text{definition red}[T] \\ =\ & g\circ(f_i\ \bar{s}) & \text{abstraction} \\ =\ & (h_i\,\bar{r})\circ g & \text{premise \& } \textbf{see below} \\ =\ & (h_i\,(E_i(\mathrm{red}[T](\bar{h}))\ \bar{x}))\circ g & \text{substitution for } \bar{r} \\ =\ & (\mathrm{red}[T](\bar{h})\ (C_i\ \bar{x}))\circ g & \text{definition red}[T] \end{array}$$

To justify the step, labeled **see below** above, we need show $(E_i(\lambda r\,.\,r\circ g)\ \bar{r})\ =\ (E_i(\lambda s\,.\,g\circ s)\ \bar{s})$ for $\bar{s}\ =\ E_i(\mathrm{red}[T](\bar{f}))\ \bar{x}$ and $\bar{r}\ =\ E_i(\mathrm{red}[T](\bar{h}))\ \bar{x}$.

$$\begin{array}{lll} & E_i(\lambda s\,.\,g\circ s)\ \bar{s} & \\ =\ & E_i(\lambda s\,.\,g\circ s)\ (E_i(\mathrm{red}[T](\bar{f}))\ \bar{x}) & \text{substitution of } \bar{s} \\ =\ & E_i((\lambda s\,.\,g\circ s)\ \circ\ (\mathrm{red}[T](\bar{f})))\ \bar{x} & E_i \text{ is a functor} \\ =\ & E_i(\lambda s\,.\,g\circ(\mathrm{red}[T](\bar{f})\ s))\ \bar{x} & \beta \text{ reduction} \\ =\ & E_i(\lambda s\,.\,(\mathrm{red}[T](\bar{h})\ s)\circ g)\ \bar{x} & \text{ind. hyp.} \\ =\ & E_i((\lambda s\,.\,s\circ g)\circ(\mathrm{red}[T](\bar{h})))\ \bar{x} & \text{inverse } \beta \text{ reduction} \\ =\ & E_i(\lambda s\,.\,s\circ g)(E_i(\mathrm{red}[T](\bar{h}))\ \bar{x}) & E_i \text{ is a functor} \\ =\ & E_i(\lambda s\,.\,s\circ g)\ \bar{r} & \text{substitution of } \bar{r} \\ =\ & E_i(\lambda r\,.\,r\circ g)\ \bar{r} & \alpha \text{ renaming} \end{array}$$

For example for $T\ =\ \mathit{list}$ the general formula is instantiated as:

$$\frac{\begin{array}{c}(E_{Nil}(\lambda r\,.\,r\circ g)\ ())\ =\ (E_{nil}(\lambda s\,.\,g\circ s)\ ())\ \ \Rightarrow\ \ (h_{nil}())\circ g\ =\ g\circ(f_{Nil}())\ \wedge \\ (E_{Cons}(\lambda r\,.\,r\circ g)\ (r,rs))\ =\ (E_{Cons}(\lambda s\,.\,g\circ s)\ (s,ss))\ \ \Rightarrow\ \ (h_{Cons}(r,rs))\circ g\ =\ g\circ(f_{Cons}(s,ss))\end{array}}{g\circ(\mathrm{red}[\mathit{list}](\bar{f})\,x)\ =\ (\mathrm{red}[\mathit{list}](\bar{h})\,x)\circ g}$$

Reducing the expressions involving E_{Nil} and E_{Cons}.

$$\frac{\begin{array}{c}()\ =\ ()\Rightarrow\ \ h_{nil}()\circ g\ =\ g\circ f_{Nil}()\ \wedge \\ (r,rs\circ g)\ =\ (s,g\circ ss)\Rightarrow\ \ h_{Cons}(r,rs)\circ g\ =\ g\circ f_{Cons}(s,ss)\end{array}}{g\circ(\mathrm{red}[\mathit{list}](\bar{f})\,x)\ =\ (\mathrm{red}[\mathit{list}](\bar{h})\,x)\circ g}$$

Applying the theorem $(x,y)\ =\ (a,b)\ \ \Leftrightarrow\ \ (x\ =\ a)\wedge(y\ =\ b)$.

$$\frac{\begin{array}{c}h_{nil}()\circ g\ =\ g\circ f_{nil}() \\ (r\ =\ s)\ \ \wedge\ \ (rs\circ g)=(g\circ ss)\ \ \Rightarrow\ \ h_{Cons}(r,rs)\circ g\ =\ g\circ f_{Cons}(s,ss)\end{array}}{g\circ(\mathrm{red}[\mathit{list}](\bar{f})\,x)\ =\ (\mathrm{red}[\mathit{list}](\bar{h})\,x)\circ g}$$

An extension of the normalization algorithm based upon this theorem as well which given the f_i computes the h_i is what is needed. To this end we propose the following. Let there be two new kinds of terms $\mathbf{inv}^g\,(x)$ (as before) and $\mathbf{swap}^g(r,x)$ for use only in the algorithm. Let these terms have the property that $g\;(\mathbf{inv}^g\,(x))$ normalizes to x, and that $g\;(\mathbf{swap}^g(r,x))$ normalizes to $r(g\;x)$. Thus from the Second-order Promotion Theorem we reason:

$$\begin{array}{lcll}
(h_i\;\bar{r})\circ g & = & g\circ(f_i\;\bar{s}) & \text{from the theorem}\\
(h_i\;\bar{r}) & = & g\circ(f_i\;\bar{s})\circ\mathbf{inv}^g & \text{normalizing } \mathbf{inv}^g\\
(h_i\;\bar{r}) & = & g\circ(f_i\;(E_i(\lambda r\,.\,\lambda x\,.\,\mathbf{swap}^g(r,x))\bar{r}))\circ\mathbf{inv}^g & \text{normalizing } \mathbf{swap}^g(,)
\end{array}$$

Then we can express the computation that computes h_i as follows:

$$h_i \;=\; N[\;\lambda\bar{r}\,.\,g\circ(f_i(E_i(\lambda r\,.\,\lambda x\,.\,\mathbf{swap}^g(r,x))\bar{r}))\circ\mathbf{inv}^g\;]$$

where $N[\;\;]$ is the normalizing algorithm. For example given:

$$\mathit{itrev}(x) \;=\; \mathrm{red}[\mathit{list}](\;\lambda()\,.\,\lambda w\,.\,w,\;\;\lambda(a,r)\,.\,\lambda w\,.\,r(\mathrm{Cons}(a,w))\;)\;x\;\mathrm{Nil}$$

we may compute the fusion of the term $\mathit{len}(\mathit{itrev}\,x) \;=\; (\mathrm{red}[T](\bar{h})\,x)\;(\mathit{len}\;\mathrm{Nil})$ as follows:

$$\begin{array}{rcl}
h_{Nil} & = & \lambda()\,.\,\mathit{len}\circ(f_{Nil}(E_{Nil}(\lambda r\,.\,\lambda x\,.\,\mathbf{swap}^{\mathit{len}}(r,x))\,()))\circ\mathbf{inv}^{\mathit{len}}\\
& = & \lambda()\,.\,\mathit{len}\circ(f_{Nil}\,())\circ\mathbf{inv}^{\mathit{len}}\\
& = & \lambda()\,.\,\mathit{len}\circ(\lambda w\,.\,w)\circ\mathbf{inv}^{\mathit{len}}\\
& = & \lambda()\,.\,(\lambda w\,.\,\mathit{len}\;w)\circ\mathbf{inv}^{\mathit{len}}\\
& = & \lambda()\,.\,(\lambda w\,.\,\mathit{len}(\mathbf{inv}^{\mathit{len}}\;w))\\
& = & \lambda()\,.\,(\lambda w\,.\,w))
\end{array}$$

For h_{cons} we proceed in a similar manner:

$$\begin{array}{rcl}
h_{Cons} & = & \lambda(y,ys)\,.\,\mathit{len}\circ(f_{Cons}(E_{Cons}(\lambda r\,.\,\lambda x\,.\,\mathbf{swap}^{\mathit{len}}(r,x))\,(y,ys)))\circ\mathbf{inv}^{\mathit{len}}\\
& = & \lambda(y,ys)\,.\,\mathit{len}\circ(f_{Cons}(y,\lambda x\,.\,\mathbf{swap}^{\mathit{len}}(ys,x)))\circ\mathbf{inv}^{\mathit{len}}\\
& = & \lambda(y,ys)\,.\,\mathit{len}\circ((\lambda(a,r)\,.\,\lambda w\,.\,r(\mathrm{Cons}(a,w)))(y,\lambda x\,.\,\mathbf{swap}^{\mathit{len}}(ys,x)))\circ\mathbf{inv}^{\mathit{len}}\\
& = & \lambda(y,ys)\,.\,\mathit{len}\circ(\lambda w\,.\,(\lambda x\,.\,\mathbf{swap}^{\mathit{len}}(ys,x))(\mathrm{Cons}(y,w)))\circ\mathbf{inv}^{\mathit{len}}\\
& = & \lambda(y,ys)\,.\,\mathit{len}\circ(\lambda w\,.\,(\mathbf{swap}^{\mathit{len}}(ys,\mathrm{Cons}(y,w))))\circ\mathbf{inv}^{\mathit{len}}\\
& = & \lambda(y,ys)\,.\,(\lambda w\,.\,\mathit{len}(\mathbf{swap}^{\mathit{len}}(ys,\mathrm{Cons}(y,w))))\circ\mathbf{inv}^{\mathit{len}}\\
& = & \lambda(y,ys)\,.\,(\lambda w\,.\,ys(\mathit{len}\;\mathrm{Cons}(y,w)))\circ\mathbf{inv}^{\mathit{len}}\\
& = & \lambda(y,ys)\,.\,(\lambda w\,.\,ys(1+\mathit{len}\;w))\circ\mathbf{inv}^{\mathit{len}}\\
& = & \lambda(y,ys)\,.\,(\lambda w\,.\,ys(1+\mathit{len}\;(\mathbf{inv}^{\mathit{len}}w)))\\
& = & \lambda(y,ys)\,.\,(\lambda w\,.\,ys(1+w))
\end{array}$$

Thus we have computed the composition

$$\mathit{len}(\mathit{itrev}\;\;x) \;=\; \mathrm{red}[\mathit{list}]\;(\;\lambda()\,.\,(\lambda w\,.\,w),\;\;\lambda(y,ys)\,.\,(\lambda w\,.\,ys(1+w))\;)\;\;x\;\;0$$

7 Applicability of Algebraic Programs and Future Work

The wide spread use of algebraic programming will depend upon several factors. First the ease of expressing programs algebraically, second the generality of the optimization techniques presented here to typical programs, and third the expressiveness of algebraic programs.

Our experience programming algebraically is that it is no harder to program using combinators, than it is using recursion. In fact for some applications it is easier since the combinators encode typical control patterns and relieve the programmers of tedious detail. Unusual control patterns often must be cast as co-combinators and our experience here is more limited. For some applications, algorithms can actually be constructed which are independent of their data structures. This has important implications for reusability.

The optimizations techniques are widely applicable, and we are currently investigating several other techniques which would make them even more so. Our experience with the second order promotion theorem suggests that there are many generic theorems which can be exploited in a similar fashion.

The language presented in this paper is limited in that algebraic programs can only traverse a single data structure at a time. This makes algorithms as simple natural number subtraction resort to arcane tricks or be simply unencodable. We would like to encode simple algorithms such as equality, unification, zip, and the nth element of a list function algebraically, and for these functions to be amenable to automatic transformations. Elsewhere[7] we report initial results on encoding these functions algebraically by generalizing the $E\$c_i$ functors and the combinators. These generalizations have promotion like theorems but while our experience with generic transformations is limited, our results so far have been quite encouraging.

8 Related Work

This work is related to Water's on series expressions [21]. His techniques apply only to traversals of linear data structures such as lists, vectors, and streams.

It is also related to Wadler's work on listlessness, and deforestation [19, 20, 8]. Deforestation works on all first order treeless terms. Treelessness is a syntactic property which guarantees that terms can be unfolded without introducing infinite regress. Wadler makes the observation that some intermediate data structures for primitive types, such as integers, booleans, etc. do not really take up space, so he developed a method to handle such terms, using a technique he calls blazing which extends the class of treeless programs. Treelessness can be applied to algebraic programs and normalizing a treeless program is one of the reasons the *inverse* exception would be raised. This is handled by the normalization algorithm by essentially skipping over the offending term. Wadlers language may encode algorithms which induct over several objects simultaneously which cannot be handled by the algebraic language described here.

Chin's work on fusion [2] extends Wadler's work on deforestation. He generalizes Wadler's techniques to all first order programs, not just treeless ones, by recognizing and skipping over terms to which his techniques do not apply in much the same manner the normalization algorithm does. His work also applies to higher order programs in general. This is accomplished by

a higher order removal phase, which first removes some higher order functions from a program. Those not removed are recognizable and are simply "skipped" over in the improvement phase. Our normalization algorithm needs no explicit higher order removal phase and as illustrated in Section 6 can actually simplify higher order functions, and invents laws on the fly that Chin's algorithm must know a priori.

Gill, Launchbury and Jones[11] describe a deforestation algorithm currently used in the Glasgow Haskell compiler. It uses two combinators *fold*[¶] and *build*[||], and a higher order theorem which relates the composition of the two. These techniques are limited since the two forms must be immediately adjacent, while the normalization algorithm will attempt to push these forms through intermediate compositions. In addition functions defined in terms of these combinators are hardwired into the compiler and there is no ability for a user to construct his own, other than through composition of existing ones.

Our implementation of ADL is the result of the influence of ideas from several areas. First, work by Malcom [14], Meijer, Fokkinga, and Paterson [15, 9], and Cockett [3, 4] which describe how to capture patterns of recursion for a large class of algebraic types in a uniform way. Many of the theorems which our transformation algorithms are based upon can be found here, second, our previous work on type reflection [13, 16], and our work on program normalization[5, 6, 17].

9 Conclusion

The formalism outlined above combining normalization, *Zr* law calculation, and the *tupling lemma* with *beta* and *eta* contraction provides a theoretical basis for calculation based transformations. We conjecture that in addition to the techniques outlined above a handful of other techniques similar in generality can be found to increase the tools effectiveness. The ADL language and the transformation tool have been implemented, and are in use here at OGI.

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